International Journal of Modern Research in Engineering & Management (IJMREM)
||Volume|| 2||Issue|| 1||Pages|| 26-36 || January 2019|| ISSN: 2581-4540

Lyapunov-type inequalities for a fractional q,ω-difference equation involving *p*-Laplacian operator

[1,]Yawen Yan, [2,]Chengmin Hou
[1,2,] (Department of Mathematics, Yanbian University, Yanji, 133002, P.R. China)

ABSTRACT

In this paper, we present new Lyapunov-type inequalities for a fractional boundary value problem of fractional q,ω-difference equation with p-Laplacian operator. The obtained inequalities are used to obtain a lower bound for the eigenvalues of corresponding equations.

KEYWORDS*: Lyapunov-type inequality; fractional derivative; eigenvalues; boundary value problem*

Date of Submission: Date, 10 January 2018 ⟺ Date of Accepted: 14. January 2019

I. INTRODUCTION

The *p*-Laplacian operator arises in different mathematical models that describe physical and natural phenomena (see, for example, [1-6]). In this paper, we present some Lyapunov-type inequalities for a fractional q,ω-difference equation with *p*-Laplacian operator. More precisely, we are interested with the nonlinear fractional boundary value problem

$$\begin{cases} {}_aD^{\beta}_{q,\omega}(\phi_p({}_aD^{\alpha}_{q,\omega}u(t)))+\chi(t)\phi_p(u(t))=0, & a<t<b, \\ u(a)=D_{q,\omega}u(a)=D_{q,\omega}u(b)=0, & {}_aD^{\alpha}_{q,\omega}u(a)={}_aD^{\alpha}_{q,\omega}u(b)=0, \end{cases} \tag{1.1}$$

where $2<\alpha\le 3, 1<\beta\le 2$, ${}_aD^{\alpha}_{q,\omega}$, ${}_aD^{\beta}_{q,\omega}$ are the Riemann-Liouville fractional derivatives of orders α,β, $\phi_p(s)=|s|^{p-1}$, $p>1$, and $\chi:[a,b]\to R$ is a continuous function. Under certain assumptions imposed on the function g, we obtain necessary conditions for the existence of nontrivial solutions to (1.1). Some applications to eigenvalue problems are also presented. For completeness, let us recall the standard Lyapunov inequality [7], which states that if u is a nontrivial solution of the problem

$$\begin{cases} u''(t)+\chi(t)u(t)=0, & a<t<b, \\ u(a)=u(b)=0, \end{cases}$$

where $a<b$ are two consecutive zeros of u, and $\chi:[a,b]\to R$ is a continuous function, then

$$\int_a^b |\chi(t)|dt > \frac{4}{b-a}. \tag{1.2}$$

Note that in order to obtain this inequality, it is supposed that a and b are two consecutive zeros of u. In our case, as it will be observed in the proof of our main result, we assume just that u is a nontrivial solution to (1.1). Inequality (1.2) is useful in various applications, including oscillation theory, stability criteria for periodic differential equations, and estimates for intervals of disconjugacy.

Several generalizations and extensions of inequality (1.2) to different boundary value problems exist in the literature. As examples, we refer to [8-13] and the references therein. The rest of this paper is organized as follows. In Section 2, we recall some basic concepts on fractional q,ω-calculus and establish some preliminary result be used in Section 3, where we state and prove our main result. In Section 4, we present some applications of the obtained Lyapunov-type inequalities to eigenvalue problem.

II. PRELIMINARIES

For the convenience of the reader, we recall some basic concepts on fractional q,ω-calculus to make easy the analysis of (1.1). For more details, we refer to [19]. Let $C[a,b]$ be the set of real-valued and continuous functions in $[a,b]$. Let $f \in C[a,b]$. We define the fractional q,ω-derivative of Riemann-Liouville type by

$$\left({}_a D_{q,\omega}^{\alpha} f\right)(x) = \begin{cases} \left({}_a I_{q,\omega}^{-\alpha} f\right)(x), & \alpha < 0, \\ f(x), & \alpha = 0, \\ \left(D_{q,\omega}^{\lceil \alpha \rceil}\, {}_a I_{q,\omega}^{\lceil \alpha \rceil - \alpha} f\right)(x), & \alpha < 0, \end{cases}$$

Where $\lceil \alpha \rceil$ denotes the smallest integer greater or equal to α.

Define a *q*-shifting operator as ${}_a\phi_q(m) = qm + (1-q)a$. For any positive integer k, we have

${}_a\phi_q^k(m) = {}_a\phi_q^{k-1}({}_a\phi_q(m))$ and ${}_a\phi_q^0(m) = m$.

We also define the new power of *q*-shifting operator as

$$(n-m)_a^{(0)} = 1,\ (n-m)_a^{(k)} = \prod_{i=0}^{k-1}(n - {}_a\phi_q^i(m)),\ k \in N \cup \{\infty\}.$$

More generally, if $\gamma \in R$, then

$$(n-m)_a^{(\gamma)} = \prod_{i=0}^{\infty} \frac{n - {}_a\phi_q^i(m)}{n - {}_a\phi_q^{i+\gamma}(m)},$$

with ${}_a\phi_q^{\gamma}(m) = q^{\gamma} m + (1-q^{\gamma})a,\ \gamma \in R$.

For any $\gamma, n \in R$, we have

$$(n - {}_{\omega_0}\phi_q(s))_{\omega_0}^{(\gamma)} = (n-c)^{\gamma} \prod_{i=0}^{\infty} \frac{1 - q^{i+1}\left(\frac{s-\omega_0}{n-\omega_0}\right)}{1 - q^{1+i+\gamma}\left(\frac{s-\omega_0}{n-\omega_0}\right)},$$

and $D_{q,\omega}(x-a)_{\omega_0}^{\alpha} = [\alpha]_q (x-a)_{\omega_0}^{(\alpha-1)}$.

Definition 2.1. *Let* $v>0$ *and* f *be a function defined on* $[a,b]$. *The fractional integration of Riemann-Liouville type is given by*

$$ {}_aI_{q,\omega}^{v}f(t)=\frac{1}{\Gamma_q(v)}\int_a^t (t-_{\omega_0}\phi_q(s))_{\omega_0}^{(v-1)}f(s)d_{q,\omega}s,\ \ v>0,t\in[a,b]. $$

Theorem 2.2. [19] *Let* $\alpha,\beta\in R^+$. *The Hahn's fractional integration has the following semi-group property*

$$ {}_aI_q^{\beta}\,{}_aI_q^{\alpha}f(x)=\left({}_aI_q^{\alpha+\beta}f\right)(x),\qquad (\omega_0<a<x<b). $$

$$ \left({}_aD_{q,\omega}^{\alpha}\,{}_aD_{q,\omega}f\right)(x)=\left({}_aD_{q,\omega}^{\alpha+1}f\right)(x),\qquad (\omega_0<a<x<b). $$

Lemma 2.3. *Let* f *be a function defined on an interval* (ω_0,b) *and* $\alpha\in R^+$. *Then the following is valid*

$$ \left({}_aD_{q,\omega}^{\alpha}\,{}_aI_{q,\omega}^{\alpha}f\right)(x)=f(x),\qquad (\omega_0<a<x<b). $$

Theorem 2.4. *Let* $\alpha\in(N-1,N]$. *Then for some constants* $c_i\in R, i=1,2,\cdots,N,$ *the following equality holds:*

$$ \left({}_aI_{q,\omega}^{\alpha}\,{}_aD_{q,\omega}^{\alpha}f\right)(x)=f(x)+c_1(x-a)_{\omega_0}^{(\alpha-1)}+c_2(x-a)_{\omega_0}^{(\alpha-2)}+\cdots+c_N(x-a)_{\omega_0}^{(\alpha-N)}. $$

Now, in order to obtain an integral formulation of (1.1), we need the following results.

Lemma 2.5. *Let* $2<\alpha\le 3$, *and* $y\in C[a,b]$. *Then the problem*

$$ \begin{cases} {}_aD_{q,\omega}^{\alpha}u(t)+y(t)=0, & a<t<b,\\ u(a)=D_{q,\omega}u(a)=D_{q,\omega}u(b)=0. \end{cases} $$

has a unique solution

$$ u(t)=\int_a^b G(t,s)y(s)d_{q,\omega}s, $$

where

$$ G(t,s)=\frac{1}{\Gamma_q(\alpha)}\begin{cases} \dfrac{\left(b-_{\omega_0}\phi_q(s)\right)^{(\alpha-2)}}{(b-a)^{(\alpha-2)}}(t-a)_{\omega_0}^{(\alpha-1)}, & a\le t\le s\le b,\\ \dfrac{\left(b-_{\omega_0}\phi_q(s)\right)^{(\alpha-2)}}{(b-a)^{(\alpha-2)}}(t-a)_{\omega_0}^{(\alpha-1)}-(t-_{\omega_0}\phi_q(s))_{\omega_0}^{(\alpha-1)}, & a\le s\le t\le b. \end{cases} $$

Proof *from* Theorem 2.4 we have

$$ u(t)=-{}_aI_{q,\omega}^{\alpha}y(t)+c_1(t-a)_{\omega_0}^{(\alpha-1)}+c_2(t-a)_{\omega_0}^{(\alpha-2)}+c_3(t-a)_{\omega_0}^{(\alpha-3)}. $$

The condition $u(a)=0$ implies that $c_3=0$. Therefore,

$$
\begin{aligned}
D_{q,\omega}u(t) &= -\left({}_aI_{q,\omega}^{\alpha-1}y(t)\right)+c_1[\alpha-1]_q(t-a)_{\omega_0}^{(\alpha-2)}+c_2[\alpha-2]_q(t-a)_{\omega_0}^{(\alpha-3)} \\
&= -\frac{1}{\Gamma_q(\alpha-1)}\int_a^t(t-{}_{\omega_0}\phi_q(s))_{\omega_0}^{(\alpha-2)}y(s)d_{q,\omega}\left({}_{\omega_0}\phi_q(s)\right) \\
&\quad +c_1[\alpha-1]_q\left(t-a\right)_{\omega_0}^{(\alpha-2)}+c_2[\alpha-2]_q\left(t-a\right)_{\omega_0}^{(\alpha-3)}.
\end{aligned}
$$

The condition $D_{q,\omega}u(a)=0$ implies that $c_2=0$. Then

$$
\begin{aligned}
D_{q,\omega}u(b) = &-\frac{1}{\Gamma_q(\alpha-1)}\int_a^b(b-{}_{\omega_0}\phi_q(s))_{\omega_0}^{(\alpha-2)}y(s)d_{q,\omega}s \\
&+c_1[\alpha-1]_q(b-a)_{\omega_0}^{(\alpha-2)}.
\end{aligned}
$$

Since $D_{q,\omega}u(b)=0$, we get

$$
c_1=\frac{1}{\Gamma_q(\alpha)(b-a)_{\omega_0}^{(\alpha-2)}}\int_a^b(b-{}_{\omega_0}\phi_q(s))_{\omega_0}^{(\alpha-2)}y(s)d_{q,\omega}s.
$$

Thus,

$$
\begin{aligned}
u(t) = &-\frac{1}{\Gamma_q(\alpha)}\int_a^t(t-{}_{\omega_0}\phi_q(s))_{\omega_0}^{(\alpha-1)}y(s)d_{q,\omega}s \\
&+\frac{(t-a)_{\omega_0}^{(\alpha-1)}}{\Gamma_q(\alpha)(b-a)_{\omega_0}^{(\alpha-2)}}\int_a^b(b-{}_{\omega_0}\phi_q(s))_{\omega_0}^{(\alpha-2)}y(s)d_{q,\omega}s.
\end{aligned}
$$

For the uniqueness, suppose that u_1 and u_2 are two solutions of the considered problem. Define $u=u_1-u_2$. By linearity, u solves the boundary value problem.

$$
\begin{cases}
{}_aD_{q,\omega}^{\alpha}u(t)=0, & a<t<b, \\
u(a)=D_{q,\omega}u(a)=D_{q,\omega}u(b)=0.
\end{cases}
$$

which has a unique solution $u=0$. Therefore, $u_1=u_2$ and the uniqueness follows. □

Lemma 2.6. *Let* $y\in C[a,b]$, $2<\alpha\le 3$, $1<\beta\le 2$, $p>1$, *and* $\frac{1}{p}+\frac{1}{g}=1$. *Then the problem*

$$
\begin{cases}
{}_aD_{q,\omega}^{\beta}(\phi_p({}_aD_{q,\omega}^{\alpha}u(t)))+y(t)=0, & a<t<b, \\
u(a)=D_{q,\omega}u(a)=D_{q,\omega}u(b)=0, & {}_aD_{q,\omega}^{\alpha}u(a)={}_aD_{q,\omega}^{\alpha}u(b)=0.
\end{cases}
\quad (1.1)
$$

has a unique solution

$$u(t) = -\int_a^b G(t,s)\phi_g\left(\int_a^b H(s,\tau)y(\tau)d_{q,\omega}\tau\right)d_{q,\omega}s\,,$$

$$H(t,s) = \frac{1}{\Gamma_q(\beta)}\begin{cases} \dfrac{\left(b -_{\omega_0} \phi_q(s)\right)_{\omega_0}^{(\beta-1)}}{(b-a)_{\omega_0}^{(\beta-1)}}(t-a)_{\omega_0}^{(\beta-1)}, & a \le t \le s \le b, \\ \dfrac{\left(b -_{\omega_0} \phi_q(s)\right)_{\omega_0}^{(\beta-1)}}{(b-a)_{\omega_0}^{(\beta-1)}}(t-a)_{\omega_0}^{(\beta-1)} - (t -_{\omega_0} \phi_q(s))_{\omega_0}^{(\beta-1)}, & a \le s \le t \le b. \end{cases}$$

Proof

from Theorem 2.4 we have

$$\phi_p\left({}_aD_{q,\omega}^{\alpha}u(t)\right) = -{}_aI_{q,\omega}^{\beta}y(t) + c_1(t-a)_{\omega_0}^{(\beta-1)} + c_2(t-a)_{\omega_0}^{(\beta-2)}\,,$$

where $c_i, i = 1,2$, are real constants.

The condition ${}_aD_{q,\omega}^{\alpha}u(a) = 0$ implies that $\phi_p\left({}_aD_{q,\omega}^{\alpha}u(a)\right) = 0$. which yields $c_2 = 0$. The conditions ${}_aD_{q,\omega}^{\alpha}u(b) = 0$ implies that $\phi_p\left({}_aD_{q,\omega}^{\alpha}u(b)\right) = 0$. which yields

$$c_1 = \frac{1}{(b-a)_{\omega_0}^{(\beta\text{-}1)}\Gamma_q(\beta)}\int_a^b (b -_{\omega_0} \phi_q(s))_{\omega_0}^{(\beta\text{-}1)}y(s)d_{q,\omega}s\,.$$

Therefore,

$$\phi_p\left({}_aD_{q,\omega}^{\alpha}u(t)\right) = -\frac{1}{\Gamma_q(\beta)}\int_a^t (t -_{\omega_0} \phi_q(s))_{\omega_0}^{(\beta\text{-}1)}y(s)d_{q,\omega}s$$

$$+\frac{(t-a)_{\omega_0}^{(\beta\text{-}1)}}{(b-a)_{\omega_0}^{(\beta\text{-}1)}\Gamma_q(\beta)}\int_a^b (b -_{\omega_0} \phi_q(s))_{\omega_0}^{(\beta\text{-}1)}y(s)d_{q,\omega}s\,,$$

that is,

$$\phi_p\left({}_aD_{q,\omega}^{\alpha}u(t)\right) = \int_a^b H(t,s)y(s)d_{q,\omega}s\,.$$

Then we have

$${}_aD_{q,\omega}^{\alpha}u(t) - \phi_g\left(\int_a^b H(t,s)y(s)d_{q,\omega}s\right) = 0\,.$$

Setting

$$\tilde{y} = -\phi_g\left(\int_a^b H(t,s)y(s)d_{q,\omega}s\right).$$

We obtain

$$\begin{cases} {}_aD_{q,\omega}^{\alpha}u(t) + \tilde{y}(t) = 0, & a < t < b, \\ u(a) = D_{q,\omega}u(a) = D_{q,\omega}u(b) = 0. \end{cases}$$

Finally, we applying Theorem2.4, we obtain the desired result. □

The following estimates will be useful later.

Lemma 2.7. *We have*

$$0 \le G(t,s) \le G({}_{\omega_0}\phi_q(s),s), \qquad (t,s) \in [a,b]\times[a,b].$$

Proof From the respect of $G(t,s)$, set

$$g_1(t,s) = \frac{\left(b - {}_{\omega_0}\phi_q(s)\right)^{(\alpha-2)}}{(b-a)^{(\alpha-2)}}(t-a)_{\omega_0}^{(\alpha-1)} \quad , \quad a \le t \le {}_{\omega_0}\phi_q(s) \le b\,,$$

and

$$g_2(t,s) = \frac{\left(b - {}_{\omega_0}\phi_q(s)\right)^{(\alpha-2)}}{(b-a)^{(\alpha-2)}}\cdot(t-a)_{\omega_0}^{(\alpha-1)} - (t - {}_{\omega_0}\phi_q(s))_{\omega_0}^{(\alpha-1)}\,,\ a \le {}_{\omega_0}\phi_q(s) \le t \le b\,.$$

Clearly,

$$g_1(t,s) \ge 0, \qquad a \le t \le {}_{\omega_0}\phi_q(s) \le b\,.$$

On the other hand,

$$\frac{\left(b - {}_{\omega_0}\phi_q(s)\right)^{(\alpha-2)}}{(b-a)^{(\alpha-2)}}(t-a)_{\omega_0}^{(\alpha-1)} - (t - {}_{\omega_0}\phi_q(s))_{\omega_0}^{(\alpha-1)} \ge 0 \quad , a \le s \le t \le b,$$

which yields

$$g_2(t,s) \ge 0\,,\ a \le s \le t \le b\,.$$

So $G(t,s) \ge 0$ for all $(t,s) \in [a,b]\times[a,b]$, which yields

$$0 \le G(t,s) \le G({}_{\omega_0}\phi_q(s),s), \qquad (t,s) \in [a,b]\times[a,b]\,.$$

The proof is complete. □

Lemma 2.8. *We have*

$$0 \le H(t,s) \le H({}_{\omega_0}\phi_q(s),s), \qquad (t,s) \in [a,b]\times[a,b]\,.$$

Proof

$${}_tD_{q,\omega}G(t,s) = \frac{[\alpha-1]_q}{\Gamma_q(\alpha)}\begin{cases} \dfrac{\left(b - {}_{\omega_0}\phi_q(s)\right)^{(\alpha-2)}}{(b-a)^{(\alpha-2)}}(t-a)_{\omega_0}^{(\alpha-2)}, & a \le t \le s \le b, \\ \dfrac{\left(b - {}_{\omega_0}\phi_q(s)\right)^{(\alpha-2)}}{(b-a)^{(\alpha-2)}}(t-a)_{\omega_0}^{(\alpha-2)} - (t - {}_{\omega_0}\phi_q(s))_{\omega_0}^{(\alpha-2)}, & a \le s \le t \le b. \end{cases}$$

Observe that $H(t,s) = G(t,s)$, for $\alpha - 2 = \beta - 1$, Then, from the proof of Lemma 2.8 we have

$$H(t,s)\geq 0,\ \ (t,s)\in[a,b]\times[a,b].$$

On the other hand, for all $s\in[a,b]$, we have

$$H(t,s)\leq H({}_{\omega_0}\phi_q(s),s).$$

Now, we are already to state and prove our main result.

III. MAIN RESULT

Our main result is following Lyapunov-type inequality.

Theorem 3.1. *Suppose that* $2<\alpha\leq 3$, $1<\beta\leq 2$, $p>1$, *and* $\chi:[a,b]\to R$ *is a continuous function. If (1.1) has a nontrivial continuous solution, then*

$$\int_a^b (b-{}_{\omega_0}\phi_q(s))_{\omega_0}^{(\beta\text{-}1)}({}_{\omega_0}\phi_q(s)-a)^{(\beta\text{-}1)}|\chi(s)|d_{q,\omega}s \geq \Gamma_q(\beta)[\Gamma_q(\alpha)]^{p-1}\frac{(b-a)_{\omega_0.}^{(\beta\text{-}1)}}{(b-a)_{\omega_0}^{(\alpha\text{-}2)}}\left(\int_a^b (b-{}_{\omega_0}\phi_q(s))_{\omega_0}^{(\alpha\text{-}2)}({}_{\omega_0}\phi_q(s)-a)_{\omega_0}^{(\alpha\text{-}1)}d_{q,\omega}s\right)^{1-p} \quad (3.1)$$

Proof We endow the set $C[a,b]$ with the Chebyshev norm $\|u\|_\infty$ given by

$$\|u\|_\infty=\max\{|u(t)|:a\leq t\leq b\},\ \ u\in C[a,b].$$

Suppose that $u\in C[a,b]$ is a nontrivial solution of (1.1). From Lemma 2.8 , Lemma 2.9 we have

$$u(t)=-\int_a^b G(t,s)\phi_g\left(\int_a^b H(s,\tau)\chi(\tau)\phi_p(u(\tau))d_{q,\omega}\tau\right)d_{q,\omega}s,\ \ t\in[a,b].$$

Let $t\in[a,b]$ be fixed. We have

$$\begin{aligned}|u(t)|&\leq\int_a^b|G(t,s)|\left|\phi_g\left(\int_a^b H(s,\tau)\chi(\tau)\phi_p(u(\tau))d_{q,\omega}\tau\right)\right|d_{q,\omega}s\\ &\leq\int_a^b|G(t,s)|\left|\int_a^b H(s,\tau)\chi(\tau)\phi_p(u(\tau))d_{q,\omega}\tau\right|^{g-1}d_{q,\omega}s\\ &\leq\int_a^b|G(t,s)|\theta(s)d_{q,\omega}s.\end{aligned}$$

where

$$\theta(s)=\left(\int_a^b|H(s,\tau)||\chi(\tau)||u(\tau)|^{p-1}d_{q,\omega}\tau\right)^{g-1},\ s\in[a,b].$$

Using Lemma 2.8 and Lemma 2.9, we obtain

$$|u(t)|\leq\|u\|_\infty\left(\int_a^b G({}_{\omega_0}\phi_q(s),s)d_{q,\omega}s\right)\left(\int_a^b H({}_{\omega_0}\phi_q(s),s)|\chi(s)|d_{q,\omega}s\right)^{g-1}.$$

Since the last inequality holds for every $t\in[a,b]$, we obtain

$$1\le\left(\int_a^b G({}_{\omega_0}\phi_q(s),s)d_{q,\omega}s\right)\left(\int_a^b H({}_{\omega_0}\phi_q(s),s)|\chi(s)|d_{q,\omega}s\right)^{g-1},$$

which yields the desired result. □

Corollary3.2. *Suppose that* $2<\alpha\le 3$, $1<\beta\le 2$, $p>1$, *and* $\chi:[a,b]\to R$ *is a continuous function. If* (1.1) *has a nontrivial continuous solution, then*

$$\int_a^b|\chi(s)|d_{q,\omega}s\ge(b-\omega_0)^{1-\beta}(\omega_0-a)^{1-\beta}\Gamma_q(\beta)[\Gamma_q(\alpha)]^{p-1}\frac{(b-a)_{\omega_0}^{(\beta\text{-}1)}}{(b-a)_{\omega_0}^{(\alpha\text{-}2)}}\times\left(\int_a^b(b-{}_{\omega_0}\phi_q(s))_{\omega_0}^{(\alpha\text{-}2)}({}_{\omega_0}\phi_q(s)-a)_{\omega_0}^{(\alpha\text{-}1)}d_{q,\omega}s\right)^{1-p} \tag{3.2}$$

Proof Let

$$\psi(s)=(b-{}_{\omega_0}\phi_q(s))_{\omega_0}^{(\beta\text{-}1)}({}_{\omega_0}\phi_q(s)-a)^{(\beta\text{-}1)}$$

$$=(b-\omega_0)^{\beta-1}\prod_{i=0}^{\infty}\frac{1-q^{i+1}(\frac{s-\omega_0}{b-\omega_0})}{1-q^{1+i+\gamma}(\frac{s-\omega_0}{b-\omega_0})}\cdot(\omega_0-a)^{\beta-1}\prod_{i=0}^{\infty}\frac{1-q^{i+1}(\frac{s-\omega_0}{a-\omega_0})}{1-q^{1+i+\gamma}(\frac{s-\omega_0}{a-\omega_0})}$$
$$\le(b-\omega_0)^{\beta-1}(\omega_0-a)^{\beta-1}$$

Observe that the function ψ has a maximum. That is,

$$\|\psi\|_\infty=(b-\omega_0)^{\beta-1}(\omega_0-a)^{\beta-1},\ s\in[a,b].$$

The desired result follows immediately from the last equality and inequality (3.1). □

For $p=2$, problem (1.1) becomes

$$\begin{cases}{}_aD_{q,\omega}^{\beta}({}_aD_{q,\omega}^{\alpha}u(t))+\chi(t)u(t)=0,\\ u(a)=D_{q,\omega}u(a)=D_{q,\omega}u(b)=0,\quad {}_aD_{q,\omega}^{\alpha}u(a)={}_aD_{q,\omega}^{\alpha}u(b)=0,\end{cases} \tag{3.3}$$

where $2<\alpha\le 3$, $1<\beta\le 2$, and $\chi:[a,b]\to R$ is a continuous function. In this case, taking $p=2$ in Theorem 3.1, we obtain the following result.

Corollary3.3. *Suppose that* $2<\alpha\le 3$, $1<\beta\le 2$, $p>1$, *and* $\chi:[a,b]\to R$ *is a continuous function. If (3.3) has a nontrivial continuous solution, then*

$$\int_a^b(b-{}_{\omega_0}\phi_q(s))_{\omega_0}^{(\beta\text{-}1)}({}_{\omega_0}\phi_q(s)-a)^{(\beta\text{-}1)}|\chi(s)|d_{q,\omega}s\ge\Gamma_q(\beta)\Gamma_q(\alpha)\frac{(b-a)_{\omega_0}^{(\beta\text{-}1)}}{(b-a)_{\omega_0}^{(\alpha\text{-}2)}}\left(\int_a^b(b-{}_{\omega_0}\phi_q(s))_{\omega_0}^{(\alpha\text{-}2)}({}_{\omega_0}\phi_q(s)-a)_{\omega_0}^{(\alpha\text{-}1)}d_{q,\omega}s\right)^{-1}.$$

Taking $p=2$ in Corollary3.2, we obtain the following result.

Corollary3.4. *Suppose that* $2<\alpha\le 3$, $1<\beta\le 2$, $p>1$, *and* $\chi:[a,b]\to R$ *is a continuous function. If (3.3) has a nontrivial continuous solution, then*

$$\int_a^b |\chi(s)| d_{q,\omega}s \ge (b-\omega_0)^{\beta-1}(\omega_0-a)^{\beta-1}\Gamma_q(\beta)\Gamma_q(\alpha)\frac{(b-a)_{\omega_0}^{(\beta\text{-}1)}}{(b-a)_{\omega_0}^{(\alpha\text{-}2)}} \times\left(\int_a^b (b-{}_{\omega_0}\phi_q(s))_{\omega_0}^{(\alpha\text{-}2)}({}_{\omega_0}\phi_q(s)-a)_{\omega_0}^{(\alpha\text{-}1)}d_{q,\omega}s\right)^{-1}.$$

IV. APPLICATIONS TO EIGENVALUE PROBLEMS

In this section, we present some applications of the obtained results to eigenvalue problems.

Corollary 4.1. *Let* λ *be an eigenvalue of the problem*

$$\begin{cases} {}_aD_{q,\omega}^{\beta}(\phi_p({}_aD_{q,\omega}^{\alpha}u(t)))+\lambda\phi_p(u(t))=0, & 0<t<1, \\ u(0)=D_{q,\omega}u(0)=D_{q,\omega}u(1)=0, & {}_aD_{q,\omega}^{\alpha}u(0)={}_aD_{q,\omega}^{\alpha}u(1)=0, \end{cases} \quad (4.1)$$

where $2<\alpha\le 3$, $1<\beta\le 2$, *and* $p>1$. *Then*

$$|\lambda| \ge \frac{\Gamma_q(2\beta)}{\Gamma_q(\beta)}\left(\frac{\Gamma_q(\alpha)\Gamma_q(\alpha+1)}{\Gamma_q(\alpha-1)}\right)^{p-1}. \quad (4.2)$$

Proof Let λ be an eigenvalue of (4.1). Then there exists a nontrivial solution $u=u_\lambda$ to (4.1). Using

Theorem 3.1 with $(a,b)=(0,1)$ and $\chi(s)=\lambda$, we obtain

$$|\lambda|\int_0^1 (1-{}_{\omega_0}\phi_q(s))_{\omega_0.}^{(\beta\text{-}1)}({}_{\omega_0}\phi_q(s))^{(\beta\text{-}1)}d_{q,\omega}s \ge \Gamma_q(\beta)[\Gamma_q(\alpha)]^{p-1}\frac{(1-0)_{\omega_0.}^{(\beta\text{-}1)}}{(1-0)_{\omega_0.}^{(\alpha\text{-}2)}}\left(\int_0^1 (1-{}_{\omega_0}\phi_q(s))_{\omega_0.}^{(\alpha\text{-}2)}({}_{\omega_0}\phi_q(s))_{\omega_0.}^{(\alpha\text{-}1)}d_{q,\omega}s\right)^{1-p}.$$

Observe that

$$\int_0^1 (1-{}_{\omega_0}\phi_q(s))_{\omega_0.}^{(\beta\text{-}1)}({}_{\omega_0}\phi_q(s))^{(\beta\text{-}1)}d_{q,\omega}s = B_q(\beta,\beta),$$

and

$$\int_0^1 (1-{}_{\omega_0}\phi_q(s))_{\omega_0.}^{(\alpha\text{-}2)}({}_{\omega_0}\phi_q(s))_{\omega_0.}^{(\alpha\text{-}1)}d_{q,\omega}s = B_q(\alpha\text{-}1,\alpha),$$

Where B_q is the beta function defined by?

$$B_q(x,y)=\int_0^1 \left({}_{\omega_0}\phi_q(s)\right)^{x-1}\left(1-{}_{\omega_0}\phi_q(s)\right)^{y-1}d_{q,\omega}s, \quad x,y>0.$$

Using the identity

$$B_q(x,y)=\frac{\Gamma_q(x)\Gamma_q(y)}{\Gamma_q(x+y)},$$

We get the desired result. □

Corollary 4.2. *Let λ be an eigenvalue of the problem*

$$\begin{cases} {}_aD^{\beta}_{q,\omega}({}_aD^{\alpha}_{q,\omega}u(t))+\lambda u(t)=0, & 0<t<1 \\ u(0)=D_{q,\omega}u(0)=D_{q,\omega}u(1)=0, & {}_aD^{\alpha}_{q,\omega}u(0)={}_aD^{\alpha}_{q,\omega}u(1)=0, \end{cases}$$

where $2<\alpha\le 3$, $1<\beta\le 2$*, and* $p=2$*. Then*

$$|\lambda|\ge \frac{\Gamma_q(2\beta)}{\Gamma_q(\beta)}\frac{\Gamma_q(\alpha)\Gamma_q(\alpha+1)}{\Gamma_q(\alpha-1)}.$$

Proof It follows from inequality (4.2) by taking $p=2$.

REFERENCE

1. Bognàr, G, Rontó, M: Numerical-analytic investigation of the radially symmetric solutions for some nonlinear PDEs.Comput. Math. Appl. 50, 983-991 (2005)
2. Glowinski, R, Rappaz, J: Approximation of a nonlinear elliptic problem arising in a non-Newtonian fluid flow model in glaciology. Math. Model. Numer. Anal. 37(1), 175-186 (2003)
3. Kawohl, B: On a family of torsional creep problems. J. Reine Angew. Math. 410, 1-22 (1990)
4. Liu, C: Weak solutions for a viscous p-Laplacian equation. Electron. J. Differ. Equ. 2003, 63 (2003)
5. Oruganti, S, Shi, J, Shivaji, R: Diffusive logistic equation with constant yield harvesting, I: steady-states. Trans. Am.Math. Soc. 354(9), 3601-3619 (2002)
6. Ramaswamy, M, Shivaji, R: Multiple positive solutions for classes of p-Laplacian equations. Differ. Integral Equ.17(11-12), 1255-1261 (2004)
7. Liapunov, AM: Probleme general de la stabilite du mouvement. Ann. Fac. Sci. Univ. Toulouse 2,203-407 (1907)
8. Aktas, MF: Lyapunov-type inequalities for a certain class of n-dimensional quasilinear systems.Electron. J. Differ. Equ.2013, 67 (2013)
9. Cakmak, D: On Lyapunov-type inequality for a class of nonlinear systems. Math. Inequal. Appl.16, 101 一 108 (2013)
10. Liapunov, AM: Problème général de la stabilité du mouvement. Ann. Fac. Sci. Univ. Toulouse 2, 203-407 (1907)
11. Aktas, MF: Lyapunov-type inequalities for a certain class of n-dimensional quasilinear systems. Electron. J. Differ. Equ.2013, 67 (2013)
12. Çakmak, D: On Lyapunov-type inequality for a class of nonlinear systems. Math. Inequal. Appl. 16, 101-108 (2013)
13. Hartman, P, Wintner, A: On an oscillation criterion of Liapounoff. Am. J. Math. 73, 885-890 (1951)
14. Arifi, Nassir Al, et al. "Lyapunov-type inequalities for a fractional *p*-Laplacian equation."Journal of Inequalities&Applications2016.1(2016):189.
15. Cabrera, I, Sadarangani, K, Samet, B: Hartman-Wintner-type inequalities for a class of nonlocal fractional boundaryvalue problems. Math. Methods Appl. Sci. (2016). doi:10.1002/mma.3972
16. Jleli, M, Samet, B: A Lyapunov-type inequality for a fractional *q*-difference boundary value problem. J. Nonlinear Sci.Appl. 9, 1965 一 1976 (2016)

17. O'Regan, D, Samet, B: Lyapunov-type inequalities for a class of fractional differential equations. J. Inequal. Appl. 2015,247 (2015)
18. Malinowska A B, Torres D F M. The Hahn Quantum Variational Calculus. J. Journal of Optimization Theory & Applications, 2010, 147(3):419-442.
19. Yizhu Wang,Yiding Liu,Chengmin Hou. New concepts of fractional Hahn's q,ω -derivative of Riemann-Louville type and Caputo type and applications. J. Advances in Difference Equations. December 2018, 2018:292